Name

Professor

Subject

Date

Calculus

Question No. 1

Given $y = tan\left(cos^{-1}(e^{4x})\right)$ find y' that is $\frac{dy}{dx}$. And simplify your answer.

Solution

$$y = tan\left(cos^{-1}(e^{4x})\right)$$

Using chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \;\ldots\ldots\ldots\ldots\; (1)$$

Consider $u = \left(cos^{-1}(e^{4x})\right)$ *and* $y = tan(u)$

$$\frac{dy}{du} = \frac{d}{du}\left(tan(u)\right) = \sec^2(u) \quad \ldots\ldots\ldots \; (2)$$

Substituting back the value of "u" in equation 2 gives

$$\frac{dy}{du} = \sec^2\left(\left(arc\;cos\;\;(e^{4x})\right)\right) \;\ldots\ldots\ldots (3)$$

$$\frac{du}{dx} = \frac{d}{dx}\left(\left(arc\;cos\;\;(e^{4x})\right)\right)$$

Using product rule in equation 1

$$= \frac{d}{dx}(\arccos(e^{4x}))\frac{d}{dx}\;(e^{4x})$$

As we know that $\frac{d}{dx}\arccos(\theta) = \left(-\frac{1}{\sqrt{1-(\theta)^2}}\right)$

$$\frac{du}{dx} = \left(-\frac{1}{\sqrt{1-(e^{4x})^2}}\right) \times 4(e^{4x}) = -\frac{4e^{4x}}{\sqrt{1-(e^{4x})^2}} \quad \ldots\ldots(4)$$

Substituting equation 3 and 4 in equation 1 gives

$$y' = \frac{dy}{dx} = \sec^2\left(\left(arc\ cos\ \ (e^{4x})\right)\right)\left(-\frac{4e^{4x}}{\sqrt{1-(e^{4x})^2}}\right) \quad \ldots\ldots..(5)$$

Simplification:

Using the mathematical rule $\sec\left(\left(arc\ cos\ \ (x)\right)\right) = \frac{1}{x}$ in equation 5 gives

$$\sec^2\left(\left(arc\ cos\ \ (e^{4x})\right)\right) = \left(\frac{1}{e^{4x}}\right)^2$$

Simplified form becomes

$$y' = \left(\frac{1}{e^{4x}}\right)^2\left(-\frac{4e^{4x}}{\sqrt{1-(e^{4x})^2}}\right)$$

$$y' = -\frac{4}{e^{4x}\sqrt{1-(e^{8x})}}$$

Question No. 2

Evaluate the following

$$\int_0^{\frac{\pi}{2}} \frac{\boldsymbol{sin\,x}}{\boldsymbol{sinx + cos\,x}}\,\boldsymbol{dx} = \frac{\boldsymbol{\pi}}{\boldsymbol{4}}$$

Solution

Let

$$I = \int_0^{\frac{\pi}{2}} \frac{sin\,x}{sinx + cos\,x}\,dx \quad \ldots\ \ldots\ \ldots\ \ldots(1)$$

Multiplying dominator and denominator with factor $(\sin x - \cos x)$ in equation 1.

$$I = \int_0^{\frac{\pi}{2}} \frac{sin\,x\,(\sin x - \cos x)}{(sinx + \cos x)(\sin x - \cos x)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{sin\,x\,(\sin x - \cos x)}{(\sin^2 x - \cos^2 x)}\,dx$$

As we know that

$$(\sin^2 x - \cos^2 x) = \cos\,(2x)$$

It becomes

$$(\cos^2 x - \sin^2 x) = -\cos(2x) \quad \dots \dots \dots \dots \; (2)$$

$$I = \int_0^{\frac{\pi}{2}} \frac{sin\,x\,(\sin x - \cos x)}{(\sin^2 x - \cos^2 x)}\,dx = \int_0^{\frac{\pi}{2}} \frac{-\,sin\,x\cos x + \sin^2 x}{(\sin^2 x - \cos^2 x)}\,dx \dots \dots \dots (3)$$

Inserting equation 2 in equation 3

$$I = \int_0^{\frac{\pi}{2}} \left[\frac{-\,sin\,x\cos x}{-\cos(2x)} + \frac{\sin^2 x}{\cos(2x)}\right] dx$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{sin\,x\cos x}{\cos\,(2x)} + \frac{\sin^2 x}{\cos\,(2x)}\right] dx \quad \dots \dots \dots \dots \; (4)$$

$$I = I_1 + I_2$$

Inserting equation 5 in equation 4

$$sin\,x\cos x = \frac{\sin\,(2x)}{2} \quad \dots \dots \dots \dots \; (5)$$

$$I_1 = \int_0^{\frac{\pi}{2}} \left[\frac{sin\,x\cos x}{\cos\,(2x)}\right] dx = \int_0^{\frac{\pi}{2}} \left[\frac{\frac{\sin\,(2x)}{2}}{\cos\,(2x)}\right] dx$$

$$I_1 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{\sin\ (2x)}{\cos\ (2x)}\right] dx$$

$$Substituting \quad u = \cos(2x)$$

$$I_1 = \frac{1}{2}\int -\frac{1}{2u}\, du = -\frac{1}{4}\int \frac{1}{u}\, du = -\frac{1}{4}\ln|u| \quad \ldots\ldots\ldots (6)$$

Substituting back in equation 6

$$I_1 = -\frac{1}{4}\ln|\cos(2x)| \quad \ldots\ldots\ldots (6)$$

$$I_2 = \int_0^{\frac{\pi}{2}} \left[\frac{\sin^2\ x}{\cos\ (2x)}\right] dx \quad \ldots\ \ldots\ \ldots\ldots \quad (7)$$

Using the following identity

$$\sin^2 x = \frac{1 -\ 2\cos(2x)}{2}$$

Equation 7 becomes

$$I_2 = \int_0^{\frac{\pi}{2}} \left[\frac{\frac{1 -\ 2\cos(2x)}{2}}{\cos(2x)}\right] dx$$

$$I_2 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{1 -\ 2\cos\ (2x)}{\cos\ (2x)}\right] dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{1}{\cos\ (2x)} - \frac{\cos\ (2x)}{\cos\ (2x)}\right] dx$$

$$I_2 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{1}{\cos\ (2x)} - 1\right] dx$$

$$I_2 = \frac{1}{2}\left[\int \left[\frac{1}{\cos(2x)}\right] dx - \int [1]\ dx\right]$$

$$I_2 = \frac{1}{2}\left[\left(\frac{1}{2}\ln|\tan(2x)| + \sec(2x)\right) - x\right] \ldots\ldots\ldots (8)$$

Using equation 6 and equation 8

$$I = -\frac{1}{4}\ln|\cos(2x)| + \frac{1}{2}\left[\left(\frac{1}{2}\ln|\tan(2x)| + \sec(2x)\right) - x\right]$$

Computing the boundaries

$$\lim_{x \to 0^+}\left(-\frac{1}{4}\ln|\cos(2x)| + \frac{1}{2}\left[\left(\frac{1}{2}\ln|\tan(2x)| + \sec(2x)\right) - x\right]\right) = 0$$

$$\lim_{x \to \frac{\pi}{2}^-}\left(-\frac{1}{4}\ln|\cos(2x)| + \frac{1}{2}\left[\left(\frac{1}{2}\ln|\tan(2x)| + \sec(2x)\right) - x\right]\right) = \frac{\pi}{4}$$

$$I = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Hence proved

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx = \frac{\pi}{4}$$

Question No. 3

Evaluate the following limit. Use limit theorem not $\varepsilon - \delta$ techniques. If any of them fail to exist, say so and say why.

$$\lim_{x \to 1^+} \frac{\sqrt{x-1}}{x^2 - 1}$$

Solution

First simplifying the equation by factoring $x^2 - 1 = (x+1)(x-1)$ and inserting in the equation.

$$\frac{\sqrt{x-1}}{x^2 - 1} = \frac{\sqrt{x-1}}{(x+1)(x-1)} = \frac{1}{(x+1)\sqrt{x-1}}$$

Using x approaching to 1 from the right side the condition becomes

$$x > 1 \quad \rightarrow \quad (x+1)\sqrt{x-1} > \ 0$$

The denominator from the right is a positive quantity.

$$\lim_{x \to 1^+} \frac{\sqrt{x-1}}{x^2 \ - \ 1} \ = \ \infty$$

Graph: